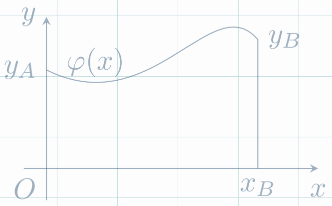

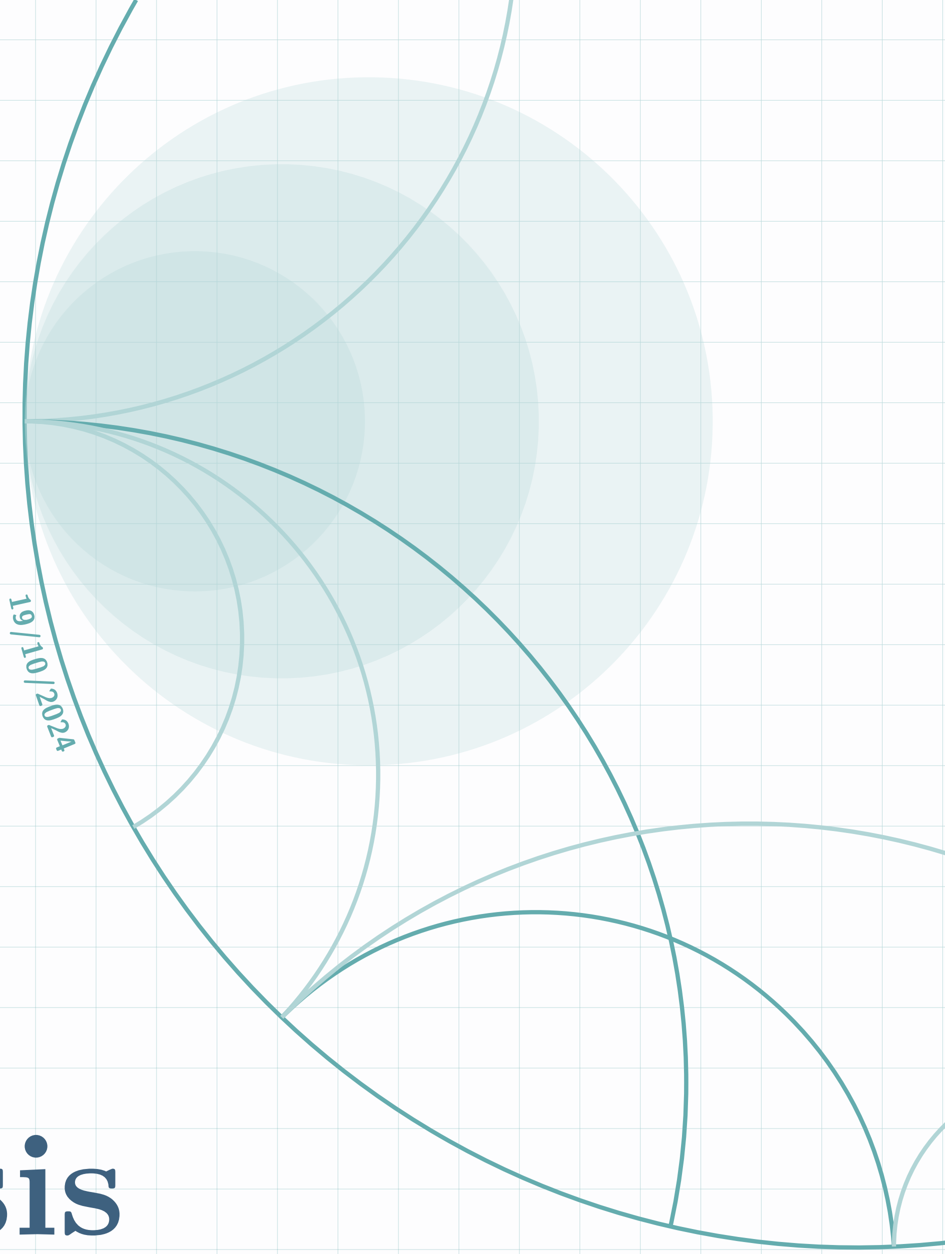

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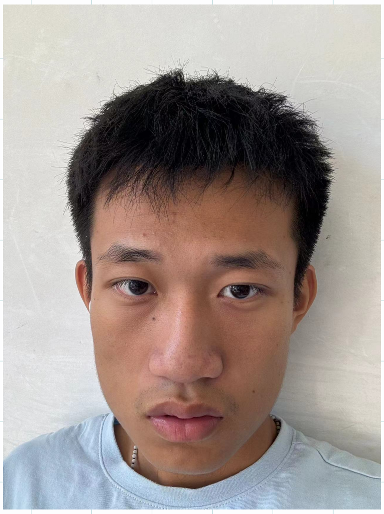
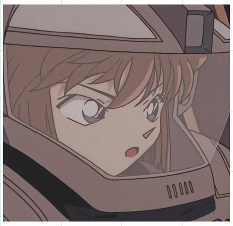


Analysis

Mathematical Analysis

(First Edition)

Sherr1



$$\begin{aligned} i\hbar\partial_t\psi(\boldsymbol{r},t) &= H\psi(\boldsymbol{r},t) \\ &= \left[-\frac{\hbar^2}{2m}\nabla^2+U(\boldsymbol{r})\right]\psi(\boldsymbol{r},t) \end{aligned}$$

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1 Curve Integral and Surface Integral

1.1 Curve Integral

1.1.1 Curve Integral (I)

Formula 1.1 Curve Integral (I)

$L \in \mathbb{R}^2$

$$L : \begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad t \in [a, b]$$

where $f(x, y) \in C(L)$ and $x(t), y(t) \in C^1[a, b]$

then

$$\int_L f(x, y)ds = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}dt$$

while $ds = \sqrt{(x'(t))^2 + (y'(t))^2}dt$

Further more,

$\Gamma \in \mathbb{R}^3$

$$L : \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \quad t \in [a, b]$$

where $f(x, y, z) \in C(L)$ and $x(t), y(t), z(t) \in C^1[a, b]$

then

$$\int_L f(x, y, z)ds = \int_a^b f(x(t), y(t), z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt$$

At this case, $ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt$

Remark For Curve Integral, it satisfies the **linear arithmetic** and the **additivity of the integral region**. On the other hand, it's holding the **inheriting order property** and the property of **absolute value inequalities**. Last but no least, It's also keeping the **mean value theorem of the integral**. ☐

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1.1.2 Curve Integral (II)

Formula 1.2 Curve Integral (II)-1

$L=\widetilde{AB}\in\mathbb{R}^2$ is a simple smooth curve,

$$L:\begin{cases}x=x(t)\\ y=y(t)\end{cases}\quad t\in I$$

Differently from **Formula 1.1**, $t=a$ corresponds to the starting point of the curve L, $t=b$ corresponds to the ending point of the curve L. And the direction of change of t from a to b is consistent with L. $f(x,y)\in C(L)$ and $x(t),y(t)\in C^1(I)$

then

$$\int_L f(x,y)dx=\int_a^b f(x(t),y(t))x'(t)dt$$

$$\int_L f(x,y)dy=\int_a^b f(x(t),y(t))y'(t)dt$$

Example 1.1

This is a specific case of **Formula 1.2**.

when $y=y(x)$, then

$$\int_{\widetilde{AB}} f(x,y)dx=\int_a^b f(x,y(x))dx$$

$$\int_{\widetilde{AB}} f(x,y)dy=\int_a^b f(x,y(x))y'(x)dx$$

Remark There is something we need to pay attention to. If the direction of change of the parameter is **not consistent with** the direction of extension of the curve, the value we calculate needs to take a **negative sign**. ☐

Formula 1.3 Curve Integral (II)-2

Based on **Formula 1.2**, when $L = \overset{\frown}{AB} \in \mathbb{R}^3$ is a simple smooth curve,

$$L : \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \quad t \in I$$

Similarly to **Formula 1.2**, $t = a$ corresponds to the starting point of the curve L, $t = b$ corresponds to the ending point of the curve L. And the direction of change of t from a to b is consistent with L. $f(x, y, z) \in C(L)$ and $x(t), y(t), z(t) \in C^1(I)$

then

$$\int_L f(x,y)dx = \int_a^b f(x(t), y(t))x'(t)dt$$

$$\int_L f(x,y)dy = \int_a^b f(x(t), y(t))y'(t)dt$$

$$\int_L f(x,y)dz = \int_a^b f(x(t), y(t))z'(t)dt$$

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1.1.3 The connection between Curve Integral (I) and (II)

Theorem 1.1

$L = \widetilde{AB} \in \mathbb{R}^2$ is a simple smooth curve, $\vec{\tau}$ is the unit tangent vector in the same direction as the curve L, $\vec{\tau} = (\cos(\vec{\tau}, x), \cos(\vec{\tau}, y))$.

If $P(x, y)$ and $Q(x, y)$ are all in $C(L)$, then

$$\int_L P(x,y)dx = \int_L P(x,y)\cos(\vec{\tau}, x)ds$$

$$\int_L Q(x,y)dy = \int_L Q(x,y)\cos(\vec{\tau}, y)ds$$

where

$$\cos(\vec{\tau}, x) = \frac{x'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}}$$

$$\cos(\vec{\tau}, y) = \frac{y'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}}$$

then

$$\cos(\vec{\tau}, x)ds = \frac{x'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2} \cdot \sqrt{(x'(t))^2 + (y'(t))^2}}dt = dx$$

$$\cos(\vec{\tau}, y)ds = \frac{y'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2} \cdot \sqrt{(x'(t))^2 + (y'(t))^2}}dt = dy$$

Further more, in $\mathbb{R}^3$ we have

$$\int_L Pdx + Qdy + Rdz = \int_L (P\cos(\vec{\tau}, x) + Q\cos(\vec{\tau}, y) + R\cos(\vec{\tau}, z))ds$$

Remark **Theorem 1.1** is the preparation of **Green Formula Gauss Formula** and **Stokes Formula**. $\square$

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1.2 Surface Integral

1.2.1 Surface Integral (I)

Formula 1.4 Surface Integral (I)-1

$f(x, y, z) \in C(S)$is a simple and smooth function on surface S,

$$S:\ z = z(x, y) \quad (x, y) \in \sigma_{xy}$$

where σ_{xy} is the area where the surface S is projected on the plane $x - O - y$ and the projected area is determined, $z(x, y) \in C^1(\sigma_{xy})$, then

$$\iint_S f(x, y, z)dS = \iint_{\sigma_{xy}} f(x, y, z(x, y))\sqrt{(z'_x)^2 + (z'_y)^2 + 1}dxdy$$

Formula 1.5 Surface Integral (I)-2

$f(x, y, z) \in C(S)$is a simple and smooth function on surface S,

$$S := \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases} \quad (u, v) \in D$$

then

$$\iint_S f(x, y, z)dS = \iint_D f(x(u, v), y(u, v), z(u, v))\sqrt{EG - F^2}dudv$$

where

$$E = (x'_u)^2 + (y'_u)^2 + (z'_u)^2 \quad G = (x'_v)^2 + (y'_v)^2 + (z'_v)^2 \quad F = x'_u x'_v + y'_u y'_v + z'_u z'_v$$

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1.2.2 Surface Integral (II)

Formula 1.6 Surface Integral (II)

S is a simple and smooth surface,

$$S := \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} \quad (u,v) \in D$$

we denote

$$A = \frac{D(y,z)}{D(u,v)} \quad B = \frac{D(z,x)}{D(u,v)} \quad C = \frac{D(x,y)}{D(u,v)}$$

then (A,B,C) is corresponding to the normal direction of S, and there are two cases

①The direction of the unit normal vector $\vec{n}$ of S is at the same direction as (A,B,C), then we have

$$\cos\alpha = \frac{A^2}{\sqrt{A^2+B^2+C^2}}$$

$$\cos\beta = \frac{B^2}{\sqrt{A^2+B^2+C^2}}$$

$$\cos\gamma = \frac{C^2}{\sqrt{A^2+B^2+C^2}}$$

then

$$\iint_S P(x,y,z)dydz = \iint_S P(x,y,z)\cos\alpha dS = \iint_D P(x(u,v),y(u,v),z(u,v))Adudv$$

$$\iint_S Q(x,y,z)dzdx = \iint_S Q(x,y,z)\cos\beta dS = \iint_D Q(x(u,v),y(u,v),z(u,v))Bdudv$$

$$\iint_S R(x,y,z)dxdy = \iint_S P(x,y,z)\cos\gamma dS = \iint_D R(x(u,v),y(u,v),z(u,v))Cdudv$$

②The direction of the unit normal vector $\vec{n}$ of S is at the opposite direction as (A,B,C), which means the direction of the unit normal vector $\vec{m}$ of $-S$ is at the opposite direction as (A,B,C), then we have

$$\iint_S P(x,y,z)dydz = -\iint_{-S} P(x,y,z)dydz = -\iint_D P(x(u,v),y(u,v),z(u,v))Adudv$$

$$\iint_S Q(x,y,z)dzdx = -\iint_{-S} Q(x,y,z)dzdx = -\iint_D Q(x(u,v),y(u,v),z(u,v))Bdudv$$

$$\iint_S R(x,y,z)dxdy = -\iint_{-S} R(x,y,z)dxdy = -\iint_D R(x(u,v),y(u,v),z(u,v))Cdudv$$

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Example 1.2

This is a specific example of **Formula 1.6**.

when

$$S:\ z = z(x,y) \quad (x,y) \in \sigma_{xy}$$

where σ_{xy} is the area where the surface S is projected on the plane $x-O-y$ and the projected area is determined, $z(x,y) \in C^1(\sigma_{xy})$, then

$$(A,B,C) = (-z'_x, -z'_y, 1)$$

so that

$$\iint_S P(x,y,z)dydz = \iint_{\sigma_{xy}} -z'_x(x,y)P(x,y,z(x,y))dydz$$

$$\iint_S Q(x,y,z)dzdx = \iint_{\sigma_{xy}} -z'_y(x,y)Q(x,y,z(x,y))dzdx$$

$$\iint_S R(x,y,z)dxdy = \iint_{\sigma_{xy}} R(x,y,z(x,y))dxdy$$

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1.3 Green/Gauss/Stokes Formulas

1.3.1 Green Formula

Theorem 1.2 Green Formula

$D \in \mathbb{R}^2$, ∂D is a directional curve composed of simple smooth closed curves with finite segments, and the direction is positive. $P(x,y), Q(x,y), \dfrac{\partial P}{\partial y}(x,y), \dfrac{\partial Q}{\partial x}(x,y) \in C(D)$, then

$$\int_{\partial D} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Actually, we have

$$\int_{\partial D} Pdx = -\iint_D \frac{\partial P}{\partial y} dxdy \quad \int_{\partial D} Qdy = \iint_D \frac{\partial Q}{\partial x} dxdy$$

$\vec{\tau}$ is the unit tangent vector in the same direction as the curve L, $\vec{\tau} = (\cos(\vec{\tau}, x), \cos(\vec{\tau}, y))$. then

$$\int_{\partial D} (P\cos(\vec{\tau}, x) + Q\cos(\vec{\tau}, y))\, ds = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

If we take $\vec{n}$ as the unit external normal vector of the curve ∂D, then

$$\cos(\vec{\tau}, x) = -\cos(\vec{n}, y) \quad \cos(\vec{\tau}, y) = \cos(\vec{n}, x)$$

$$\int_{\partial D} (P\cos(\vec{n}, x) + Q\cos(\vec{n}, y))\, ds = \int_{\partial D} (P\cos(\vec{\tau}, y) - Q\cos(\vec{\tau}, x))\, ds$$

then the Green Formula can change into

$$\int_{\partial D} (P\cos(\vec{n}, x) + Q\cos(\vec{n}, y))\, ds = \iint_D \left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dxdy$$

Proposition 1.1 area of region

If we denote $P = -y, Q = 0$ or $Q = x, y = 0$, then we have

$$V_J(D) = \iint_D dxdy = \int_{\partial D} xdy = \int_{\partial D} -ydx$$

$V_J(D)$ is the area of the region D.

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1.3.2 Gauss Formula

Theorem 1.3 Gauss Formula

$\Omega \in \mathbb{R}^3$, $\partial\Omega$ is a directional curve composed of simple smooth closed surfaces with finite segments, and the direction is to the Ω outside.
$P(x,y,z), Q(x,y,z), R(x,y,z), \dfrac{\partial P}{\partial x}(x,y,z), \dfrac{\partial Q}{\partial y}(x,y,z), \dfrac{\partial R}{\partial z}(x,y,z) \in C(\Omega)$, then

$$\iint_{\partial\Omega} Pdydz + Qdzdx + Rdxdy = \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz$$

Actually, we have

$$\iint_{\partial\Omega} Pdydz = \iiint_{\Omega} \frac{\partial P}{\partial x} dxdydz$$

$$\iint_{\partial\Omega} Qdzdx = \iiint_{\Omega} \frac{\partial Q}{\partial y} dxdydz$$

$$\iint_{\partial\Omega} Rdxdy = \iiint_{\Omega} \frac{\partial R}{\partial z} dxdydz$$

If we take $\vec{n}$ as the unit external normal vector of the surface $\partial\Omega$

$$\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$$

then we have

$$\iint_{\partial\Omega} (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS = \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz$$

Proposition 1.2 area of volume

If we denote $P = x, y = 0, z = 0$ or $x = 0, Q = y, z = 0$ or $x = 0, y = 0, R = z$, then we have

$$\begin{aligned} V_J(\Omega) &= \iiint_{\Omega} dxdy = \int_{\partial\Omega} xdydz = \int_{\partial\Omega} ydzdx = \int_{\partial\Omega} zdxdy \\ &= \frac{1}{3}\left(\int_{\partial\Omega} xdydz + \int_{\partial\Omega} ydzdx + \int_{\partial\Omega} zdxdy\right) \end{aligned}$$

$V_J(\Omega)$ is the area of the volume Ω.

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1.3.3 Stokes Formula

Theorem 1.4 Stokes Formula

The directional surface S is a smooth surface with edges, and ∂S is composed of a finite number of smooth connected closed curves and the direction of S and the direction of ∂S conform to the right-hand rule

If the first partial derivative of $P(x,y,z), Q(x,y,z), R(x,y,z)$ are continuous on S, then

$$\int_{\partial S} Pdx + Qdy + Rdz = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$$= \iint_S \begin{vmatrix} dydz & dzdx & dxdy \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

If we take $\vec{n}$ as the unit external normal vector of S

$$\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$$

then

$$\int_{\partial S} Pdx + Qdy + Rdz = \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

1.4 Integration is path independent

Theorem 1.5 Integration is path independent (I)

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